TRINITY COLLEGE

Mathematics Specialist
Test 5 2018

Section 1 Calculator Free
Implicit Differentiation, Differential Equations

STUDENT'S NAME SOLUTIONS

DATE: Friday 10 August **TIME:** 20 minutes **MARKS:** 18

INSTRUCTIONS:
Standard Items: Pens, pencils, drawing templates, eraser

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (4 marks)

Solve the differential equation $\dfrac{dy}{dx}=\dfrac{-0.5x^2}{y}$ given $x=0$ when $y=2$.

$$y\,dy = -0.5x^2\,dx$$

$$\frac{y^2}{2} = -\frac{x^3}{6} + c$$

$x=0$
$y=2$

$$2 = 0 + c$$

$$\frac{y^2}{2} = -\frac{x^3}{6} + 2$$

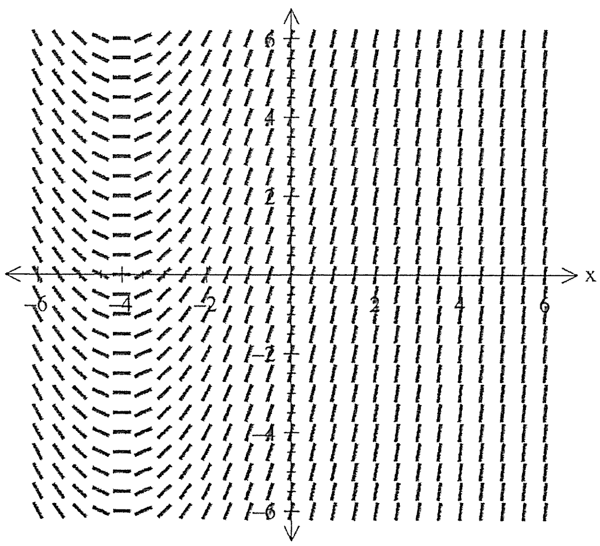
2. (8 marks)

(a) From the seven differential equations given below, match four of them with the slope fields drawn. Enter results in the table below. [4]

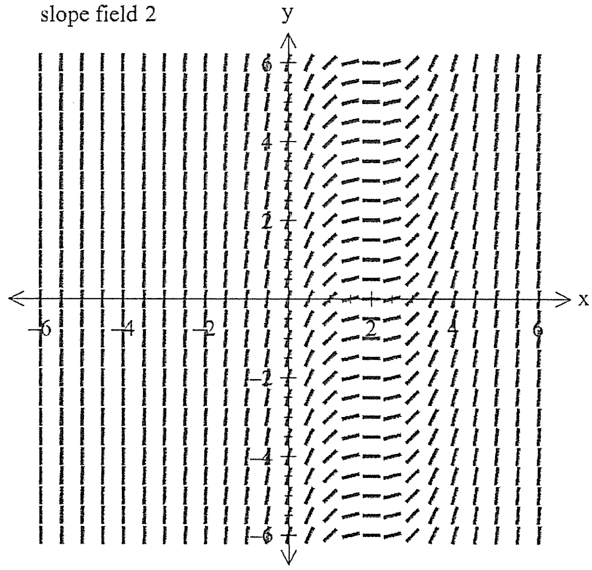
A: $y' = x + 4$ B: $y' = -\dfrac{x}{y}$ C: $y' = \sqrt{x}$ D: $y' = (x+1)(x-3)$

E: $y' = (x+3)(x-1)$ F: $y' = (x-2)^2$ G: $y' = \dfrac{x}{y}$

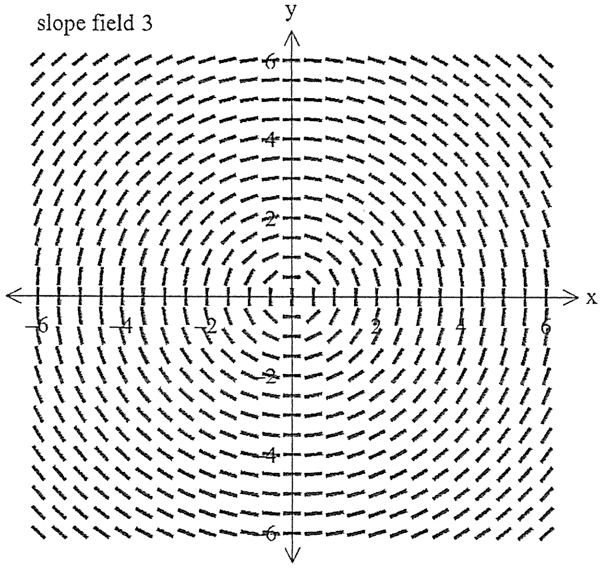
slope field 1



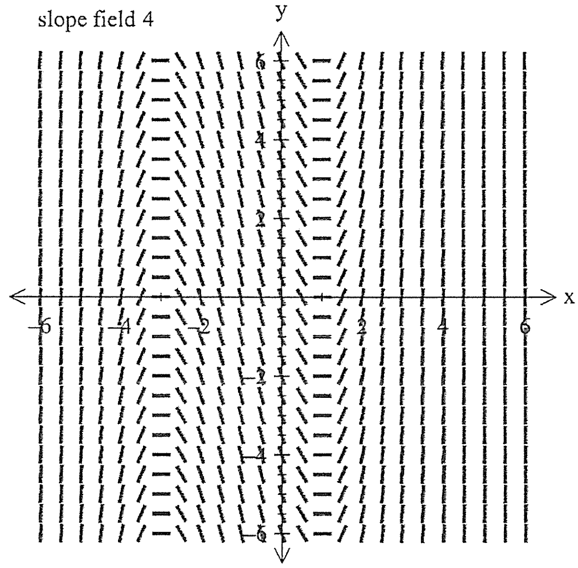
slope field 2



slope field 3



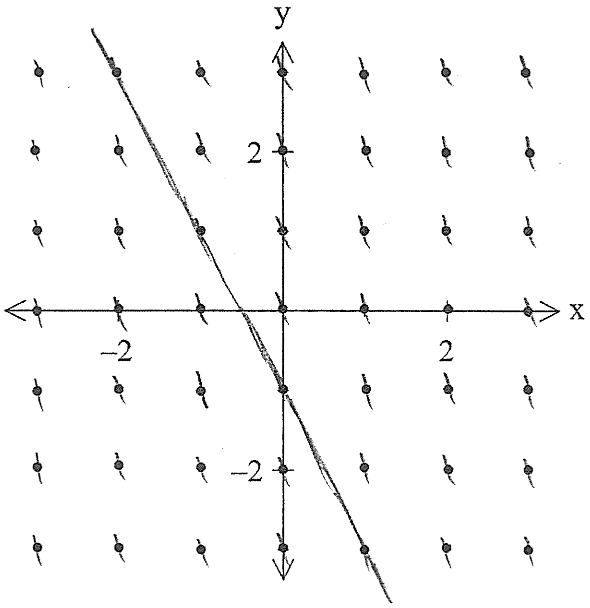
slope field 4



Slope field	1	2	3	4
Equation	A	F	B	E

(b) For the differential equation $\frac{dy}{dx} = -2$

(i) Sketch the slope field [2]



(ii) Use your slope field to sketch a particular solution through the point (1,-3) [2]

3. (6 marks)

Solve the differential equation $\frac{dp}{dq} = 2pq(p+3)$ to give a general solution.

$$\frac{dp}{p(p+3)} = 2q\,dq$$

$$\frac{A}{p} + \frac{B}{p+3} = \frac{1}{p(p+3)}$$

$$\frac{Ap + 3A + Bp}{p(p+3)} = \frac{1}{p(p+3)}$$

$$3A = 1 \qquad A + B = 0$$

$$A = \frac{1}{3} \qquad B = -\frac{1}{3}$$

$$\int \frac{1}{3p} - \frac{1}{3(p+3)} = \int 2q\,dq$$

$$\frac{1}{3}\ln p - \frac{1}{3}\ln(p+3) = q^2 + c_1$$

$$\frac{1}{3}\ln\frac{p}{p+3} = q^2 + c_1$$

$$\ln\frac{p}{p+3} = 3q^2 + c_2$$

$$\frac{p}{p+3} = e^{3q^2 + c_2}$$

$$p = pe^{3q^2 + c_2} + 3e^{6q + c_2}$$

$$p(1 - e^{6q + c_2}) = 3e^{3q^2 + c_2}$$

$$p = \frac{3e^{3q^2 + c_2}}{1 - e^{3q^2 + c_2}}$$

$$p = \frac{Ae^{3q^2}}{1 - Be^{3q^2}}$$

A, B CONSTANT

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Section 1 Calculator Free
Implicit Differentiation, Differential Equations

STUDENT'S NAME ______________________________

DATE: Friday 10 August **TIME:** 30 minutes **MARKS:** 30

INSTRUCTIONS:

Standard Items:	Pens, pencils, drawing templates, eraser
Special Items:	Three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment)

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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4. (6 marks)

Elephant population on a reserve had been reduced by poaching to only 200 before a very strict anti-poaching policy allowed the elephants to recover. The population, P, increased according to the logistics model $\frac{dP}{dt} = 0.096P - 0.000016P^2$ where t is in years.

(a) Determine the maximum elephant population the reserve can sustain. [1]

$$\frac{dP}{dt} = 0.000016P(6000 - P)$$

$\therefore$ 6000 ELEPHANTS

(b) Write an equation for the elephant population. [2]

$$P = \frac{6000 \times 200}{200 + 5800e^{-0.000016 \times 6000t}}$$

$$= \frac{1200000}{200 + 5800e^{-0.096t}} \qquad \left(\frac{6000}{1 + 29e^{-0.096t}}\right)$$

(c) What is the rate of increase of the elephant population when the population reaches 1000? [1]

80 ELEPHANTS/YR

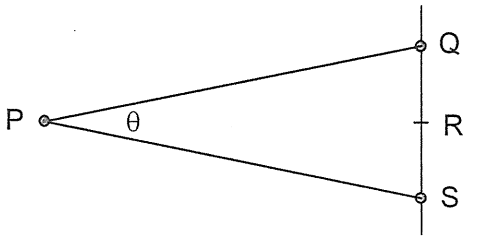
(d) How long will it take to reach 3000 elephants? [2]

$$3000 = \frac{1200000}{200 + 5800e^{-0.096t}}$$

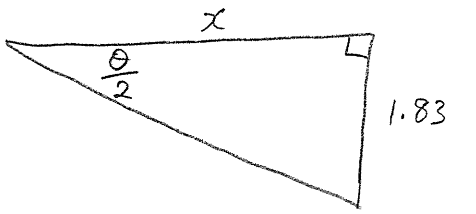
$t \approx 35$ YRS

5. (6 marks)

The diagram shows a hockey player at P running directly towards R, the midpoint of QS, where Q and S are the goalposts spaced 3.66 m apart at one end of a hockey pitch. PR is perpendicular to QS and θ, the shooting angle, is the size of angle QPS.



If the player is running at a constant speed of 6 m/s towards R, at what rate is the shooting angle θ increasing at the instant when the player is 9 m from R?



$$\tan\frac{\theta}{2} = \frac{1.83}{x}$$

diff w.r.t. t

$$\frac{1}{2\cos^2\frac{\theta}{2}}\frac{d\theta}{dt} = \frac{-1.83}{x^2}\frac{dx}{dt}$$

$$\tan\frac{\theta}{2} = \frac{1.83}{9}$$

$$\frac{\theta}{2} = 0.2$$

$$x = 9$$

$$\frac{dx}{dt} = -6$$

$$\frac{d\theta}{dt} = 0.26/\text{sec}$$

OR $14.92^\circ/\text{sec}$

6. (9 marks)

Two variable resistors with resistance M Ohms and N Ohms respectively are connected in parallel so that the Total Resistance R Ohms is given by $\dfrac{1}{R} = \dfrac{1}{M} + \dfrac{1}{N}$.

(a) Use implicit differentiation to write a differential equation linking $\dfrac{dR}{dt}$, $\dfrac{dM}{dt}$ and $\dfrac{dN}{dt}$ [2]

diff wrt t

$$-\frac{1}{R^2}\frac{dR}{dt} = -\frac{1}{M^2}\frac{dM}{dt} - \frac{1}{N^2}\frac{dN}{dt}$$

(b) At the instant when $M = 50$ Ohms and $N = 200$ Ohms, M is increasing at a rate of 10 Ohms per minute.

(i) Determine R at this instant. [1]

40

(ii) Use Calculus methods to determine the rate of change of N (in Ohms per minute), at this instant, if R is increasing at a rate of 5 Ohms per minute. Show clearly how you obtained your answer. [2]

$$-\frac{1}{1600} \times 5 = -\frac{1}{2500} \times 10 - \frac{1}{40000} \times \frac{dN}{dt}$$

$$\frac{dN}{dt} = -35$$

(c) Given that $M = N^2$, use the increments formula to calculate the approximate change in R when N changes from 50 Ohms to 51 Ohms. [4]

$$\frac{dM}{dN} = 2N$$

$$\frac{1}{R} = \frac{1}{N^2} + \frac{1}{N}$$

$$\frac{1}{R} = \frac{1+N}{N^2}$$

$$R = \frac{N^2}{1+N}$$

$$\frac{dR}{dN} = \frac{2N(1+N) - N^2}{(1+N)^2}$$

$$= \frac{2N + 2N^2 - N^2}{(1+N)^2}$$

$$= \frac{2N + N^2}{(1+N)^2}$$

$$\delta R \approx \frac{dR}{dN} \times \delta N$$

$$\approx \frac{100 + 2500}{51^2} \times 1$$

$$= 1$$

7. (9 marks)

A chemist places a lump of metal, initially at a temperature of 24°C into a hot research oven. The rate of change of temperature of the metal can be modelled by $\dfrac{dT}{dt} = k(450 - T)$ where T is the temperature in °C, t minutes after being placed in the oven and k is a positive constant. After 20 seconds, the temperature of the metal bar has risen by 39°.

(a) Show all steps to turn the given differential equation into the formula for T in terms of t and state the value of k. [5]

$$\int \frac{1}{450 - T}\, dT = \int k\, dt$$

$$-\ln(450 - T) = kt + c$$

$$450 - T = Ae^{-kt}$$

$$T = 450 - Ae^{-kt}$$

$t = 0 \quad T = 24 \qquad 24 = 450 - A$

$$\therefore A = 426$$

$$T = 450 - 426e^{-kt}$$

$t = \frac{1}{3} \quad T = 63 \qquad 63 = 450 - 426e^{-\frac{k}{3}}$

$$\therefore k = 0.288$$

$$T = 450 - 426e^{-0.288t}$$

(b) What is the expected temperature of the lump of metal after 5 minutes? [1]

$349.1\ ^\circ c$

(c) When the temperature of the lump of metal is within 5° of its maximum, the power supply to the oven is cut off and no further heating occurs. After how many minutes does this occur? [3]

$$445 = 450 - 426e^{-0.288t}$$

$t = 15.43$ mins